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On a connection between Lorentzian and Euclidean metrics

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Abstract

We investigate connections between pairs of pseudo-Riemannian metrics whose sum is a (tensor) product of a covector field with itself. A bijective mapping between the classes of Euclidean and Lorentzian metrics is constructed as a special result. The existence of such maps on a differentiable manifold is discussed. Similar relations for metrics of arbitrary signature on a manifold are considered. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

In [1], the time is defined as a congruence of lines on a real differentiable manifold M. The vector field t tangent to this congruence is called *temporal* field. In the work mentioned is stated that the Maxwell equations on M with an Euclidean metric e_{ij}, $i, j = 1, \dots, n := \dim M$ are derivable from the standard electromagnetic Lagrangian on M with a pseudo-Riemannian metric $g_{ij} = t_i t_j - e_{ij}$, $t_i := e_{ij}t^j$. In the paper cited g_{ij} is said to be Lorentzian. Special metrics g_{ij} of this kind, when the norm of t is two (with respect to both metrics — see Eq. (4.1)), are considered (e.g. [2], Section 2.6; [3], p. 219; [4], p. 148, Lemma 36). A slightly more general construction of the kind mentioned can be found in [3, pp. 241–242]. For it, without an investigation, is stated that it is Lorentzian again, which is

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not always the case (see Section 2). In the above constructions t can also be taken to be the gradient vector field of the global time function [3,5].

The purpose of the present work is to investigate pairs of pseudo-Riemannian metrics (g_{ij}, h_{ij}) whose sum is a product of the covariant components of some vector field t, i.e., $g_{ij} + h_{ij} = t_i t_j$ with, e.g., $t_i := g_{ij} t^j$.[2] In particular, we prove an important result for physics that for any real Euclidean (resp. Lorentzian) metric there exists a real Lorentzian (resp. Euclidean) metric forming with it such a pair.

The general case, for arbitrary pseudo-Riemannian metric g_{ij}, is investigated in Section 2. If g_{ij} has a signature (p, q), i.e., if the matrix $[g_{ij}]$ has p positive and q negative eigenvalues,[3] then the signature of h_{ij}, if it is non-degenerate, which is the conventional case, can be (q, p) or $(q + 1, p - 1)$. As a side result, we prove that if g_{ij} is an Euclidean metric, then (for $g_{ij} t^i t^j \neq 1$) the metric $h_{ij} = t_i t_j - g_{ij}$ can be only Lorentzian or negatively definite. As a corollary, we construct a map from the set of Euclidean metrics into the set of Lorentzian ones. The applicability of the results of Section 2 is studied in Section 3. Here we point to some topological obstacles that may arise in this direction. Section 4 is devoted to some mappings between classes of Riemannian metrics and their properties. We construct bijective maps from the set of metrics with signature (p, q) on the ones with signature $(q + 1, p - 1)$, which, in particular, is valid for the classes of Euclidean and Lorentzian metrics.[4] Bijective real maps between pseudo-Riemannian metrics of arbitrary signature are also constructed. Some concluding remarks are presented in Section 5. We also correct some wrong statements of [1].

Now, to fix the terminology, which significantly differs in different works, we present some definitions.

Following [8, p. 273], we call Riemannian metric on a real differentiable manifold M a non-degenerate, symmetric and 2-covariant tensor field g on it. If for any non-zero vector v at $x \in M$ is fulfilled $g_x(v, v) > 0$, the metric is called proper Riemannian, positive definite, or Euclidean; otherwise it is called indefinite or pseudo-Riemannian [8,9]. It is known that every finite-dimensional paracompact differentiable manifold admits positively definite (Euclidean) metrics ([8], p. 280; [9], Chapter IV, Section 1; Chapter I, Example 5.7; [10], Chapter 1, Excercise 2.3). A pseudo-Riemannian metric with exactly one positive eigenvalue is called Lorentzian [2] (or some times Minkowskian).[5] If in the above defini-

[2] Bundle decompositions and correspondences between various types of metric tensors are consequences of the Witt (decomposition) theorem [6, Chapter XIV, Section 5]. The present paper deals with one specific such correspondence based on the use of a vector field t with appropriate properties.

[3] Some times the pair (p, q) is called type of g and the signature is defined as the number $s = p - q$. In this paper, we suppose the numbers p and q to be independent of the point at which they are calculated, i.e., here we consider metrics whose signature is point-independent and so constant over the corresponding sets. The numbers p and q are also known as positive index and (negative) index of the metric. Often, especially in the physical literature, the signature is defined as an order n-tuple $(\varepsilon_1, \dots, \varepsilon_n)$ where p (resp. $q = n - p$) of $\varepsilon_1, \dots, \varepsilon_n$ are equal to $+1$ (resp. -1) or simply to the plus (resp. minus) sign and the order of $\varepsilon_1, \dots, \varepsilon_n$ corresponds to one of the signs of the diagonal elements of the metric in some pseudo-orthogonal basis.

[4] A (partial) correspondence between Euclidean and Lorentzian metrics is established in [7] via the Einstein equations.

[5] One can also find the definition of a Lorentzian metric as a metric with only one negative eigenvalue [4, p. 55]. This definition is isomorphic to the one used in the present paper (see, e.g. [4, pp. 92–93]).

tions the non-degeneracy condition is dropped, the prefix 'semi-' is added to the names of the corresponding metrics [11], e.g., a semi-Riemannian metric on M is a symmetric two times covariant tensor field on it [11].

2. Basic results

It is said that a Riemannian metric g on $U \subseteq M$ is of signature (p, q), $p + q = n := \dim M$, if it has p positive and q negative eigenvalues. A semi-Riemannian metric on U is of signature (p, q) and defect r (or of signature (p, q, r), or r-degenerate with signature (p, q)), $p + q + r = n$, if it has p positive, q negative, and r vanishing eigenvalues.

Throughout this paper the Latin indices run from 1 to $n := \dim M < \infty$ and a summation from 1 to n over indices repeated on different levels is assumed.

Proposition 2.1. *Let g be a Riemannian metric of signature (p, q) on $U \subseteq M$, t be a vector field on U,*

$$\tilde{U}^{+}_{\substack{>\\=\\<}} := \{x \,|\, x \in U,\ g(t,t)|_x \gtreqqless 1\},$$

and

$$g \mapsto \tilde{g}^{+} := h := g(\cdot, t) \otimes g(\cdot, t) - g. \tag{2.1}$$

Then the tensor field h is:

(i) *a Riemannian metric with signature (q, p) on $\tilde{U}^{+}_{<}$;*
(ii) *a Riemannian metric with signature $(q + 1, p - 1)$ on $\tilde{U}^{+}_{>}$;*
(iii) *a (parabolic) semi-Riemannian metric with signature $(q, p - 1)$ and defect 1 on $\tilde{U}^{+}_{=}$, i.e., on $\tilde{U}^{+}_{=}$ the bilinear map h has q positive, $(p - 1)$ negative, and one vanishing eigenvalue.*

Proof. Since g is by definition a 2-covariant symmetric tensor field, so is h too. Hence, the eigenvalues of h remains to be studied.

Let $x \in U$ be an arbitrary fixed point. We shall prove the proposition at x, i.e., for $U = \{x\} \subset M$. Then the general result will be evident as $U = \cup_{x \in U}\{x\}$. All the quantities given in this proof will be taken at x; so their restriction at x will not be written explicitly. We shall distinguish two cases.

'Non-isotropic' case: t is non-isotropic, i.e., $g(t, t) \neq 0$ and hence $t \neq 0$. Let $\{E'_i\}$ be a basis in $T_x(M)$, the space tangent to M at x, consisting of non-isotropic vectors with $E_1 = t$. Applying to this basis the standard Gramm–Schmidt orthogonalization procedure ([12], Chapter 4, Section 3; [13], pp. 206–208), with respect to the scalar product $(\cdot, \cdot) = g(\cdot, \cdot)$, we can construct (after normalization) a pseudo-orthogonal basis $\{E_i\}$ (at x) such that $E_1 = t/\alpha, \alpha := +\sqrt{|g(t, t)|}$ and $g_{ij} := g(E_i, E_j) = \varepsilon_i \delta_{ij}$ (i is not a summation index here!), where $p \in \mathbb{N} \cup \{0\}$ of the numbers $\varepsilon_1, \ldots, \varepsilon_n$ are equal to $+1$ while the others

$q = n - p$ of them are equal to -1 and δ_{ij} are the Kroneker deltas. With respect to $\{E_i\}$, we easily obtain

$$[h_{ij}] = \mathrm{diag}(\varepsilon_1(g(t,t) - 1), -\varepsilon_2, \dots, -\varepsilon_n), \qquad \varepsilon_1 = \mathrm{sign}(g(t,t)). \tag{2.2}$$

From here the formulated results follow immediately.

'Isotropic' case: t is isotropic, i.e., $g(t,t) = 0$. As it is easily seen, this is possible only for $t = 0$ or for $n := \dim M \geq 2$ and $q, p \geq 1$ if $t \neq 0$.

Let s_1 and s_2 be vector fields on U and s_2 be orthogonal to t with respect to g, $g(s_2, t) = 0$. From (2.1) we obtain $g(s_1, t) + h(s_1, t) = 0$ and $g(s_2, s_2) + h(s_2, s_2) = 0$ which imply h to be of signature (q, p) as the one of g is (p, q). Indeed, choosing a basis $\{E_i\}$ in which the components of g and h are, respectively, $g_{ij} = g_i\delta_{ij}$, p of $g_1, \dots, g_n$ being equal to $+1$, the rest of them being equal to -1, and $h_{ij} = h_i\delta_{ij}$, $h_i = \pm 1$,[6] we get $\sum_i (g_i + h_i) s_1^i t^i = 0$, and $\sum_i (g_i + h_i)(s_2^i)^2 = 0$, where s_1^i, s_2^i and t^i are the components of, respectively, s_1, s_2, and t in $\{E_i\}$. The first equation implies $(g_i + h_i) = 0$ for $i \in I := \{j : j \in \{1, \dots, n\}, t^j \neq 0\}$ and the second one with $s_2^i = 0$ for $i \in I$ and $s_2^i = \lambda^i \in \mathbb{R}$ for $i \notin I$ gives $(g_i + h_i) = 0$ for $i \notin I$.[7] Hence we have $(g_i + h_i) = 0$ for all i which means that the signatures of g and h are opposite. □

Corollary 2.1. *Let g be a Riemannian metric of signature* (p, q) *on* $U \subseteq M$ *and t be a vector field on U. Assume t can be chosen such that* $g(t,t)$ *is less than, or greater than, or equal to one on the whole set U. Then on U the tensor field h given by* (2.1) *is:*
- (i) *a Riemannian metric with signature* (q, p) *for* $g(t,t) < 1$;
- (ii) *a Riemannian metric with signature* $(q + 1, p - 1)$ *for* $g(t,t) > 1$;
- (iii) *a (parabolic) semi-Riemannian metric with signature* $(q, p - 1)$ *and defect* 1 *for* $g(t,t) = 1$, *i.e., in this case h has q positive,* $(p - 1)$ *negative, and one vanishing eigenvalue*.

Proof. This result is a version of Proposition 2.1 corresponding to the choice of t such that one of the sets $\tilde{U}_<^+$, $\tilde{U}_>^+$, and $\tilde{U}_=^+$ is equal to U. □

It is clear that if g is a Riemannian metric on U, then, choosing arbitrary some vector field t on U with $g(t,t) > 1$, the map (2.1) yields (infinitely) many semi-Riemannian metrics on U whose signature (and, possibly, defect) depends on the norm $g(t,t)$ on U. Generally, different t generate different metrics $\tilde{g}^+$ from one and the same initial metric g.

Corollary 2.2. *Let t be a vector field over* $U \subseteq M$, *e be an Euclidean metric on U,*

$$U_{\substack{> \\ = \\ <}} := \{x | x \in U,\ e(t,t)|_x \ \substack{> \\ = \\ <}\ 1\},$$

[6] The existence of such a basis is a simple consequence of the existence of a non-degenerate (generally non-pseudo-orthogonal) transformation which reduces two square matrices to a diagonal form simultaneously (see, e.g. [12, Chapter 4, Section 12]), in particular Theorem 6 of this reference can easily be modified in such a way that to be valid for arbitrary symmetric real matrices (hint: replace the unit matrix with $\mathrm{diag}(\varepsilon_1, \dots, \varepsilon_n)$, $\varepsilon_i = \pm 1$, and use $(x, y) = \sum_i \varepsilon_i x^i y^i$ instead of $(x, y) = \sum_i x^i y^i$).

[7] Since $g(s_2, t) = \sum_i g_i s_2^i t^i = 0$, the particular choice of s_2 is admissible.

and

$$g = e(\cdot, t) \otimes e(\cdot, t) - e. \tag{2.3}$$

Then the tensor field (2.3) *is:*
(i) *a negatively definite Riemannian metric on* $U_<$;
(ii) *a Lorentzian metric on* $U_>$;
(iii) *a one-degenerate negatively definite semi-Riemannian metric on* $U_=$.

Proof. See Proposition 2.1 for $(p, q) = (n, 0)$.[8] □

Summing up, if there exists a vector field t satisfying $e(t, t) \neq 1$ or $e(t, t) = 1$ at every point, then (2.3) defines a metric on M for which there are three possibilities. First, if $e(t, t) > 1$, it is Lorentzian. Second, if $e(t, t) = 1$, it is semi-Riemannian, viz. a one-degenerate metric, and, consequently, non-Riemannian one [11], and third, if $e(t, t) < 1$, it is negatively definite, and so isomorphic to an Euclidean metric. From physical view-point, the most essential result is that if for every e we choose some vector field t_e with $e(t_e, t_e) > 1$, then the mapping $e \mapsto g$, given by (2.3) for $t = t_e$, maps the class of Euclidean metrics on M into the class of Lorentzian ones. It is clear, this mapping essentially depends on the choice of the vector fields t_e used in its construction.

One may ask, what would happen if the signs before the terms in the right-hand side of (2.1) are (independently) changed? The change of the sign before the first term results in the following assertion.

Proposition 2.2. *Let g be a Riemannian metric of signature* (p, q) *on* $U \subseteq M$, *t be a vector field on U,*

$$\tilde{U}^-_{\substack{> \\ = \\ <}} := \{x \,|\, x \in U,\ -g(t, t)|_x \gtreqqless 1\},$$

and

$$g \mapsto \tilde{g}^- := -g(\cdot, t) \otimes g(\cdot, t) - g. \tag{2.4}$$

Then $\tilde{g}^-$ *is:*
(i) *a Riemannian metric with signature* (q, p) *on* $\tilde{U}^-_<$;
(ii) *a Riemannian metric with signature* $(q - 1, p + 1)$ *on* $\tilde{U}^-_>$;
(iii) *A (parabolic) semi-Riemannian metric with signature* $(q - 1, p)$ *and defect* 1 *on* $\tilde{U}^-_=$, *i.e., on* $\tilde{U}^-_=$ *the bilinear map h has q positive,* $(p - 1)$ *negative, and one vanishing eigenvalue.*

Proof. This proof is almost identical to the one of Proposition 2.1. The only difference is that in it $g(t, t)$ must be replaced by $-g(t, t)$. Formally this proof can be obtained from the one of Proposition 2.1 by replacing in it t^i by $\mathrm{i}t^i$, $\mathrm{i} := \sqrt{-1}$. □

[8] For an independent proof, see LANL xxx archive server, E-print No. gr-qc/9802057.

The change of the sign before the second term in (2.1) and in (2.4) is equivalent to put $g = -g'$ with g' being Riemannian metric with signature (p, q). Then, since $g(t, t) = -g'(t, t)$ and the signature of g is (q, p), we obtain valid versions of Propositions 2.1 and 2.2 if we replace in them g, p, and q with $-g$, q, and p, respectively. Thus, we have proved the next result.

Proposition 2.3. *Let g be a Riemannian metric of signature (p, q) on $U \subseteq M$, t be a vector field on U,*

$$U^{\pm}_{\substack{>\\=\\<}} := \{x \mid x \in U,\ \mp g(t,t)|_x \gtreqqless 1\} = \tilde{U}^{\mp}_{\substack{>\\=\\<}},$$

and

$$g \mapsto g^{\pm} := \pm g(\cdot, t) \otimes g(\cdot, t) + g. \tag{2.5}$$

Then $g^{\pm}$ is:

(i) *a Riemannian metric with signature (p, q) on $U^{\pm}_{<}$;*
(ii) *a Riemannian metric with signature $(p \pm 1, q \mp 1)$ on $U^{\pm}_{>}$;*
(iii) *a (parabolic) semi-Riemannian metric with defect 1 and signature $(p+(\pm 1-1)/2, q+(\mp 1-1)/2)$ on $U^{\pm}_{=}$, i.e., in this case $g^{\pm}$ has $p + (\pm 1 - 1)/2$ positive, $q + (\mp 1 - 1)/2$ negative, and one vanishing eigenvalue.*

3. Applicability of the results

Up to this point we have supposed two major things: the existence of Euclidean or Riemannian metrics and of a vector field t with the corresponding properties on $U \subset M$ or on the whole manifold M. In this sense the above considerations are local or global, respectively. Different conditions for global or local existence of (Euclidean) metrics are well-known and are discussed at length in the corresponding literature (see, e.g. [14], Chapter IV or [9,15]). In our case, the existence of Euclidean metric on M is a consequence of the paracompactness and finite-dimensionality of the manifold M [9]. These assumptions are enough for the most physical applications and we assume their validity in this work. [9]

What concerns the existence of a vector field t with properties required on a manifold with Euclidean metric e ($e(t, t)$ to be greater than, or equal to, or less than 1), some problems may arise. If on t we do not impose additional restrictions, it always can be constructed as follows: take a non-vanishing vector field t_0, [10] on M so $e(t_0, t_0) \neq 0$ (everywhere on M). Defining $t := \sqrt{a}t_0/\sqrt{e(t_0, t_0)}$ for $a \in \mathbb{R}$, $a \geq 0$, we get $e(t, t) = a$. Hence, choosing $a \gtreqqless 1$, we obtain $e(t, t) \gtreqqless 1$. Obviously, the existence of t in the first two cases, $e(t, t) \geq 1$, is equivalent to the existence of a non-vanishing vector field on M, while in the last one, $e(t, t) < 1$, this is not necessary, viz. in it t may vanish on some subsets on M or even to be the zero vector field on M.

[9] See the partial discussion of this problem in [3, Section 5.2].
[10] Generally t_0 is discontinuous (vide infra).

The general conclusion is a vector field with $e(t,t) \geq 1$ (over M) exists iff M admits nowhere vanishing (on M) vector field. Thus, our results concerning the case $e(t,t) \geq 1$ are applicable iff such a field exists. As we said above, this is just the situation if we do not impose additional conditions on t. But this is not satisfactory from the view-point of concrete applications. For instance, in most mathematical investigations the (Euclidean or semi-Riemannian) metrics are required to be differentiable of class C^1 [8,9,15,16], e.g., in the Riemannian geometry one normally uses C^2 metrics. Such an assumption implies t to be of class of smoothness at least C^1. Analogous is the situation in physics, e.g., the treatment of t as a temporal field requires t to be at least continuous [1] and the considerations on the background of general relativity force us to assume t to be of class C^2 [2].

Therefore, of great importance is the case when the vector field t satisfies certain smoothness conditions, viz. when it is of class C^m for some $m \geq 0$. At this point some topological obstacles may arise for the global existence of t with $e(t,t) \geq 1$. In fact, the above-said implies that a vector field t of class C^m with $e(t,t) \geq 1$ exists on $U \subseteq M$ iff on U there exists a C^m non-vanishing vector field. But it is well-known that not every manifold admits such a tangent vector field [17]. Classical examples of this kind are the even-dimensional spheres $\mathbb{S}^{2k}$, $k \in \mathbb{N}$: on $\mathbb{S}^{2k}$ does not exist non-vanishing (on the whole $\mathbb{S}^{2k}$) continuous vector field ([17], [18, Section 4.24]). Examples of the opposite kind are the odd-dimensional spheres $\mathbb{S}^{2k-1}$ ([17], [18, Excerise 4.26]) and the path-connected manifolds with flat C^1 linear connection: they always admits global C^1 non-vanishing vector fields. [11] Also every non-compact manifold admits C^0 non-zero vector field [19]. An analysis of the question of existence of vector fields (and Lorentz metrics) can be found in [4] where also other examples are presented. Consequently, the global existence of C^m, $m \geq 0$ field t with $e(t,t) \geq 1$ depends on the concrete manifold M and has to be investigated separately for any particular case.

The situation for an arbitrary Riemannian metric g is completely the same as described above in the Euclidean case. If on t, some additional, e.g., smoothness, conditions are not imposed a vector field t on U with $g(t,t)|_U \gtreqless 1$ can always be constructed for every $U \subseteq M$. In fact, let t_0 be any (generally discontinuous) non-vanishing on U vector field. By rescaling locally the components of t_0 we can obtain from it a non-vanishing vector field t_0' such that $g(t_0', t_0')|_U \neq 0$ and $\operatorname{sign}(g(t_0', t_0')|_U) = \varepsilon =$ constant. Defining $t := \sqrt{a}\, t_0' |g(t_0', t_0')|^{-1/2}$ for $a \in \mathbb{R}$, $a \geq 0$, we get $g(t,t) = \varepsilon a$. Consequently, by an appropriate choice of ε and a, we can realize t with $g(t,t)|_U \gtreqless 1$. Since g is by definition non-degenerate (the kernel of g consists of the zero vector field on U), the relation $g(t,t)|_U \geq 1$ implies t to be a non-vanishing vector field. Obviously, this conclusion does not concern the case of t with $g(t,t)|_U < 1$ when t can vanish somewhere or everywhere on U.

The case for t with $g(t,t)|_U \geq 1$ is completely different when $U = M$ and C^m, $m \geq 0$, metrics and vector fields are considered: generally, such a vector field does not exist globally,

[11] In the last case, such a vector field can be constructed as follows. Fix a non-zero vector v_0 at an arbitrary point x_0 of a simple-connected manifold M. Define the vector field v at any $x \in M$ as the result of the parallel transport, assigned to the given flat connection, of v_0 from x_0 to x along some path connecting x_0 and x. Then v is a tangent vector field on M which is non-vanishing and of class C^1.

i.e., on the whole manifold M. This existence depends on the topological properties of M and has to be explored separately in any concrete case.

In conclusion, the results of Section 2 are valid locally and for their global, i.e., on the whole manifold M, validity may arise obstacles of pure topological nature. Since on M, due to the paracompactness and finite-dimensionality, an Euclidean metric always exists, in the Euclidean case this is connected with the existence of a vector field t with properties required.

Now consider the class of (resp. smooth) Lorentzian metrics on M, i.e., those g for which $(p, q) = (1, n-1)$. For them, according to Corollary 2.1, the metric h is of signature $(n-1, 1)$ for $g(t, t) < 1$ and $(n, 0)$ for $g(t, t) > 1$ (resp. if such t exists on M), i.e., in the former case g and h are isomorphic and in the latter one h is a Euclidean metric. Thus, if for every g we choose some vector field t_g with $g(t_g, t_g) > 1$, then the whole class of (resp. smooth) Lorentzian metrics is mapped into the class of (resp. smooth) Euclidean ones by the mapping $g \mapsto h$ given by (2.1) for $t = t_g$ (resp. if such smooth t_g exists on M). Evidently, different vector fields t_g realize different such maps.

4. On properties of some mappings between Riemannian metrics

Some natural questions are in order. Let G^U (resp. $G^U_{p,q}$) be the set of all Riemannian metrics (resp. of signature (p, q)) on $U \subseteq M$. If t is a fixed vector field on U, then what is the character of the map $\varphi^t_U : G^U \to G^U$ given by (2.1)? For instance, can it be subjective, injective, or bijective? Can any two Riemannian metrics (with 'corresponding' signatures) be mapped into each other by φ^t_U for a suitable t? etc.

Proposition 4.1. *Let $g \in G^U_{p,q}$, t be arbitrarily fixed vector field on U, and $\varphi^t_U : G^U \to G^U$ be given via* (2.1). *Then:*

(i) *the map $\varphi^t_U|_{t=0}$ is bijection*;

(ii) *if $\lambda \in \mathbb{R}\backslash\{1\}$, the map φ^t_U is injection on the set $\{g : g \in G^U_{p,q},\ g(t, t) = \lambda\}$*;

(iii) *if* $\dim M \geq 2$, *the map φ^t_U is $(\infty^{n-1} - 1)$-to-one on the set $\{g : g \in G^U_{p,q},\ g(t, t) = 1\}$. More precisely, for every g in this set there exist $g_{(1)} \in G^U_{p,q}$ and semi-Riemannian metric $g_{(2)}$ of signature $(p - 1, q)$ and defect 1 depending on $n - 1$ real parameters, which for $g_{(1)}$ are not all zeros, such that $g_{(1)} \neq g$, $g_{(2)} \neq g$, $\varphi^t_U(g) = \varphi^t_U(g_{(1)}) = \varphi^t_U(g_{(2)})$, $g_{(1)}(t, t) = 1$, $g_{(2)}(t, t) = 0$, and $g_{(1)}$ and $g_{(2)}$ are the only solutions of the equation $\varphi^t_U(g) = \varphi^t_U(g')$ with respect to the semi-Riemannian metric g'*;

(iv) *the map φ^t_U is two-to-one on the sets $\{g : g \in G^U_{p,q},\ g(t, t) \neq 0, \frac{1}{2}\}$ for $n :=$* $\dim M = 1$ *and $\{g : g \in G^U_{p,q},\ g(t, t) \neq 0, \frac{1}{2}, 1\}$ for $n \geq 2$. More precisely, for g in these sets there exists a unique semi-Riemannian metric g' such that $\varphi^t_U(g) = \varphi^t_U(g')$ and $g \neq g'$. The signature and defect of g' depend on t; in particular, for $g(t, t) < 1$ the signature of g' is (p, q) and only for $n = 1$ and $g(t, t) = 1$ the metric g' is semi-Riemannian, its defect being equal to* 1, *i.e., $g' = 0$.*

Remark 4.1. *The choice of g with $g(t, t) = \frac{1}{2}$ (resp. $g(t, t) = 0$ for $n = 1$ and $g(t, t) = 0, 1$ for $n \geq 2$) in case* (iv) *returns us to the case* (ii) *(resp. case* (iii)). *To prove this, use that* (2.1) *implies $g'(t, t) = g(t, t), 1 - g(t, t)$ if $\varphi^t_U(g) = \varphi^t_U(g')$.*

Proof. *Case* (i). For $t = 0$ we have $\varphi_U^t(g) = -g$, so φ_U^0 is reversing of the the metric [4], p. 92 and hence it is bijective.

Case (ii). Let $g^1, g^2 \in G_{p,q}^U$ and $g^a(t,t) = \lambda \neq 1$, $a = 1, 2$. We have to show that $\varphi_U^t(g^1) = \varphi_U^t(g^2)$ implies $g^1 = g^2$. Since from (2.1) follows $(\varphi_U^t(g^a))(s,t) = (\lambda - 1)g^a(s,t)$ and $(\varphi_U^t(g^a))(r,r) = -g^a(r,r)$ for arbitrary vector fields s and r on U with $g(r,t) = 0$, in a basis $\{E_i\}$ in which the matrices of both g^1 and g^2 are diagonal, $g_{ij}^a = g_i^a\delta_{ij}$ (i is not a summation index here!) with $g_i^a = \pm 1$,[12] the equation $\varphi_U^t(g^1) = \varphi_U^t(g^2)$ implies $\sum_i (g_i^1 - g_i^2)t^i s^i = 0$ and $\sum_i (g_i^1 - g_i^2)(r^i)^2 = 0$. The former of the last two equations gives $g_i^1 - g_i^2 = 0$ for $i \in I := \{j : j \in \{1, \dots, n\},\ t^j \neq 0\}$ and the latter one with $r^i = 0$ for $i \in I$ and $r^i = \lambda^i \in \mathbb{R}$ for $i \notin I$ yields $g_i^1 - g_i^2 = 0$ for $i \notin I$. Therefore $g_i^1 = g_i^2$ which is equivalent to $g^1 = g^2$ in $\{E_i\}$.

Case (iii). Suppose $g \in G_{p,q}^U$ with $g(t,t) = 1$ is given. We have to solve the equation $\varphi_U^t(g) = \varphi_U^t(g')$ with respect to $g' \in G_{p,q}^U$, $g' \neq g$. In the special basis (at some $x \in M$ and with respect to g) defined in the 'non-isotropic' case of the proof of Proposition 2.1, this equation reads

$$g'_{ij} = g_{ij} - \delta_{i1}\delta_{j1} + g'_{i1}g'_{j1}$$

as in the basis chosen $t^i = \alpha\delta^{i1}$, $g_{ij} = \varepsilon_i\delta_{ij}$ with $\varepsilon_i = \pm 1$, and $g(t,t) = 1$ implies $\varepsilon_1 = \alpha^2 = +1$. Choosing $g'_{i1} = \alpha_i \in \mathbb{R}$ for $i \geq 2$, we get $g'_{ij} = g_{ij} + \alpha_i\alpha_j$ for $i, j \geq 2$. Since $g_{11} = \varepsilon_1\delta_{11} = 1$, for $i, j = 1$ we obtain the equation $g'_{11} = (g'_{11})^2$ with solutions $g_{(1)\,11} = 1 = g_{11}$ and $g_{(2)\,11} = 0$. Consequently, in the special basis used, we find the following two solutions of $\varphi_U^t(g) = \varphi_U^t(g')$ with respect to g':

$$g_{(1)\,ij} = \begin{cases} 1 = g_{11} & \text{for } i, j = 1, \\ g_{ij} + a_i a_j & \text{for } i, j \geq 2, \\ a_i & \text{for } i \geq 2,\ j = 1, \\ a_j & \text{for } i = 1,\ j \geq 2, \end{cases} \qquad g_{(2)\,ij} = \begin{cases} 0 & \text{for } i, j = 1, \\ g_{ij} + b_i b_j & \text{for } i, j \geq 2, \\ b_i & \text{for } i \geq 2,\ j = 1, \\ b_j & \text{for } i = 1,\ j \geq 2, \end{cases}$$

where $a_i, b_i \in \mathbb{R}$. Since $g_{ij} = \varepsilon_i\delta_{ij}$ with $\varepsilon_1 = 1$ and $\varepsilon_i = \pm 1$ for $i \geq 2$, we have $g_{(2)} \neq g$ and the equality $g_{(1)} = g$ is valid iff $a_1 = \dots = a_n = 0$.

The equalities $g_{(1)}(t,t) = 1$ and $g_{(2)}(t,t) = 0$ are evident consequences of $t^i = \alpha\delta^{i1}$ with $\alpha^2 = 1$.

Since a simple calculation shows $\det[g_{(1)\,ij} - \lambda\delta_{ij}] = \det[g_{ij} - \lambda\delta_{ij}]$ and $\det[g_{(2)\,ij} - \lambda\delta_{ij}] = \det[g_{ij} - g_{11}\delta_{i1}\delta_{j1} - \lambda\delta_{ij}]$ for $\lambda \in \mathbb{R}$, the metric $g_{(1)}$ is of signature (p,q) and $g_{(2)}$ is of signature $(p-1,q)$ and defect 1 as the signature of g is (p,q).

Case (iv). For $n = 1$ and $g(t,t) = 1$, the assertion is a corollary of the proof of case (iii) above: since $g_{11} = 1$ in the basis used in it, the solutions of $\varphi_U^t(g) = \varphi_U^t(g')$ are $g_{(1)} = g$ and $g_{(2)} = 0$ and hence $g_{(2)} \neq g$. So, below we suppose $n \geq 2$ and $g(t,t) \neq 1$.

If s is an arbitrary vector field on U, applying $\varphi_U^t(g) = \varphi_U^t(g')$ to the pair (s,t) and using (2.1), we get $[g(t,t) - 1]g(s,t) = [g'(t,t) - 1]g'(s,t)$. The choice $s = t$ yields

$$g'(t,t) = 1 - g(t,t) \neq 0, \tfrac{1}{2}, 1$$

[12] See footnote 6.

as we look for g' with $g' \neq g$ and $g(t,t) \neq 0, \frac{1}{2}, 1$. Therefore, the last equation reduces to $[g(t,t)-1]g(s,t) + g(t,t)g'(s,t) = 0$. Writing this equation in a basis $\{E_i\}$ in which both metrics g and g' are diagonal, i.e., $g_{ij} = g_i\delta_{ij}$, and $g'_{ij} = g'_i\delta_{ij}$, we obtain $g'_i = ((1/g(t,t)) - 1)g_i$ for $i \in I := \{j : j \in \{1, \dots, n\}, t^j \neq 0\}$, where $t := t^i E_i$. Analogously, defining on U a vector field r with $r^i := 0$ for $i \in I$ and $r^i := \lambda^i \in \mathbb{R}$ for $i \notin I$ in $\{E_i\}$, we see that $g(r,t) = 0$ and $(\varphi^t_U(g))(r,r) = (\varphi^t_U(g'))(r,r)$ is equivalent to $g(r,r) = g'(r,r)$, which, when written in $\{E_i\}$ implies $g'_i = g_i$ for $i \notin I$ as λ^i, $i \notin I$, are completely arbitrary. Consequently, the equation $\varphi^t_U(g) = \varphi^t_U(g')$ has a unique solution with respect to g' which in the basis $\{E_i\}$ is

$$g'_{ij} = g'_i\delta_{ij}, \qquad g'_i = \begin{cases} \left(\frac{1}{g(t,t)} - 1\right) g_i & \text{for } t^i \neq 0, \\ g_i & \text{for } t^i = 0, \end{cases}$$

where t^i are the components of t in $\{E_i\}$, $t = t^i E_i$. From these results, the rest of the assertion in case (iv) follows. □

Remark 4.2. *From the proof of Proposition* 4.1 *follows that in the case* (iv) *of Proposition* 4.1, *when* φ^t_U *is* 2:1 *map,* $g'(t,t) = 1 - g(t,t)$ *is fulfilled while in the case* (iii) *is valid* $g_{(1)}(t,t) = g(t,t)$ *and* $g_{(2)}(t,t) = 1 - g(t,t)$. *These connections agree with the general relation* $g'(t,t) = g(t,t), 1 - g(t,t)$ *which is a consequence of* (2.1).

Proposition 4.2. *Let the vector field t be arbitrarily fixed on U and the map* $\varphi^t_U : G^U \to G^U$ *be given by* (2.1). *Then* $(\varphi^t_U \circ \varphi^t_U)(g) = g$ *iff* $g \in G^U$ *is such that* $g(t,t) = 0, 2$.

Remark 4.3. *Note, due to* (2.1) *, we have*

$$(\varphi^t_U(g))(t,t) = g(t,t) \qquad \text{iff } g(t,t) = 0, 2. \tag{4.1}$$

Proof. Applying (2.1), we get $(\varphi^t_U \circ \varphi^t_U)(g) = \left\{[g(t,t) - 1]^2 - 1\right\} g(\cdot,t) \otimes g(\cdot,t) + g$. Therefore $(\varphi^t_U \circ \varphi^t_U)(g) = g$ iff $g(t,t) = 0, 2$. □

From the just-proved result immediately follows (see also [20, p. 14, Proposition 6.9])

Corollary 4.1. *The map* φ^t_U *for given t is bijective on the sets* $G^U_{t;2} := \{g : g \in G^U,\ g(t,t) = 2\} \subset G^U$ *and* $G^U_{t;0} := \{g : g \in G^U,\ g(t,t) = 0\} \subset G^U$.

Remark 4.4. *The bijectiveness of* φ^t_U *on* $G^U_{t;2}$ *does not contradict to Proposition* 4.1, *case* (iv). *Actually, if* $g \in G^U_{t;2}$, $g' \in G^U$, $g \neq g'$, *and* $\varphi^t_U(g') = \varphi^t_U(g)$, *then* (*see Remark* 4.2) $g'(t,t) = 1 - g(t,t) = -1 \neq 2$, *i.e.,* g' *is not in* $G^U_{t;2}$.

We have to note that if the Riemannian metrics g and h are given, then generally there does not exist a vector field t connecting them through $h = g(\cdot,t) \otimes g(\cdot,t) - g$. There are two reasons for this. On one hand, by Proposition 2.1 for this the metrics g and h must be of 'corresponding' signature, viz. (p,q) and $(q+1, p-1)$ or (p,q) and (q,p), respectively. On the other hand, in local coordinates the mentioned connection between g and h reduces to a system of $\frac{1}{2}n(n+1)$ equations for the n components of t and, consequently, it has solution(s) only in some exceptional cases. It is clear, even for Euclidean metric g and Lorentzian metric h such t exists only as an exception, not in the general case.

Corollary 4.2. *Let $p \geq 1$, for every $g \in G^U_{p,q}$ a vector field t_g on U be chosen such that $g(t_g, t_g) = 2$, and $T := \{t_g : g \in G^U_{p,q}\}$. Then the map $\varphi^T_{p,q} : G^U_{p,q} \to G^U_{q+1,p-1}$ given via*

$$g \mapsto \tilde{g}^+ := g(\cdot, t_g) \otimes g(\cdot, t_g) - g \tag{4.2}$$

is bijective, i.e., one-to-one onto map.

Proof. At first we note that t_g with $g(t_g, t_g) = 2$ always exists for every g because of $p \geq 1$. (For example, one can set $t_g = \sqrt{2}t_0/\sqrt{g(t_0, t_0)}$, where in a basis in which $g_{ij} = \varepsilon_i \delta_{ij}$, $\varepsilon_i = \pm 1$, and $\varepsilon_k = +1$ for some fixed $k \in \{1, \dots, n\}$ the components of t_0 are $t_0^i = \alpha \delta^{ik}$, $\alpha \in \mathbb{R}\backslash\{0\}$; so then $g(t_0, t_0) = \alpha^2 > 0$.) Now, from Proposition 4.2 and Corollary 2.1, case (ii), we deduce

$$\varphi^T_{q+1,p-1} \circ \varphi^T_{p,q} = \mathrm{id}_{G^U_{p,q}}, \quad \varphi^T_{p,q} \circ \varphi^T_{q+1,p-1} = \mathrm{id}_{G^U_{q+1,p-1}}, \tag{4.3}$$

from where, by virtue of [20, p. 14, Proposition 6.9], the result formulated follows. □

Corollary 4.2 demonstrates the existence of a bijective correspondence between the classes of Riemannian metrics with signature (p, q) and $(q+1, p-1)$ on any differentiable manifold admitting such metrics (and vector fields with corresponding properties — see Section 3). The explicit dependence of this mapping on the choice of the vector fields t_g utilized in its construction has to be emphasized. In particular, this is essential for physics: there is a bijective correspondence between the sets of Euclidean and Lorentzian metrics as they have signatures $(n, 0)$ and $(1, n-1)$, respectively. [13]

From here an important result follows. Since every paracompact finite-dimensional differentiable manifold admits Euclidean metrics ([8], p. 280; [9], Chapter IV, Section 1, Chapter I, Example 5.7), *on any such manifold admitting a vector field with an Euclidean norm greater than one exist Lorentzian metrics* as they are in bijective correspondence with the Euclidean ones. [14] The opposite statement is also true: if on M exist Lorentzian, h, and Euclidean, e, metrics, then there is a vector field t with $e(t, t) > 1$. [15] In fact, since h is Lorentzian, there is exactly one positive eigenvalue λ_+, $\lambda_+ > 0$, for which the equation $h_{ij}t^j_+ = \lambda_+ e_{ij} t^j_+$ has a non-zero solution t_+ defined up to a non-zero constant multiplier. Choosing this multiplier such that $h(t_+, t_+) > \lambda_+$, we find $e(t_+, t_+) > 1$. Let us recall (see Section 3) that the existence of t with $e(t, t) > 1$ is equivalent to the one of a non-vanishing vector field on M. So, if, as usual, we admit e, h, and t to be of class C^m, $m \geq 0$, then such a vector field may not exist on the whole M. If this happens to be the case, the above, as

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[13] In the four-dimensional case, a special type of relation between Euclidean and Lorentzian metrics is established in [7] via the Einstein equations.

[14] See ([4], p.149, Proposition 37; [19]) for more general results on the existence of Lorentzian metrics.

[15] Generally h, e, and t are not connected via (2.3).

well as the following, considerations have to be restricted on the neighborhood(s) admitting non-vanishing vector field of class C^m.

Corollary 4.3. *Let for every metric $g \in G^U_{p,q}$ be chosen a vector field t_g on U such that $g(t_g, t_g) = 0$ and $T := \{t_g : g \in G^U_{p,q}\}$. Then the map $\psi^T_{p,q} : G^U_{p,q} \to G^U_{q,p}$ defined by* (4.2) *is a bijection.*

Proof. At the beginning we notice that one can always put $t_g = 0$ for every $g \in G^U_{p,q}$ but if $p, q \geq 1$, then for any $g \in G^U_{p,q}$ exists $t_g \neq 0$ with $g(t_g, t_g) \neq 0$. (In a basis in which $g_{ij} = \varepsilon_i \delta_{ij}$, $\varepsilon_i = \pm 1$ and $\varepsilon_k + \varepsilon_l = 0$ for some fixed $k, l \in \{1, \dots, n\}$ we can set $t^i_g = \alpha(\delta^{ik} + \delta^{il})$, $\alpha \in \mathbb{R}\backslash\{0\}$.) From Proposition 4.2 and Corollary 2.1, case (i), we infer

$$\psi^T_{p,q} \circ \psi^T_{q,p} = \mathrm{id}_{G^U_{q,p}}, \qquad \psi^T_{q,p} \circ \psi^T_{p,q} = \mathrm{id}_{G^U_{p,q}}, \tag{4.4}$$

which concludes the proof. □

Let us fix some bijective maps $\varphi_{p,q} : G^U_{p,q} \to G^U_{q+1,p-1}$ and $\psi_{p,q} : G^U_{p,q} \to G^U_{q,p}$ given via (4.2) for t_g with $g(t_g, t_g) = 2$ and $g(t_g, t_g) = 0$, respectively. Here $G^U_{p,q}$ is the set of Riemannian metrics on U with signature (p, q). (Let us recall that in the 'smooth' case we cannot put $U = M$ as, generally, then $\varphi_{p,q}$ may not exist.) Then the map $\chi_{p,q} := \psi_{q+1,p-1} \circ \varphi_{p,q} : G^U_{p,q} \to G^U_{p-1,q+1}$ is bijective for any $p, q \in \mathbb{N} \cup \{0\}$ such that $p + q = n := \dim M$. Hence

$$G^U_{n,0} \stackrel{\chi_{n,0}}{\to} G^U_{n-1,1} \stackrel{\chi_{n-1,1}}{\to} G^U_{n-2,2} \stackrel{\chi_{n-2,2}}{\to} \cdots \stackrel{\chi_{2,n-2}}{\to} G^U_{1,n-1} \stackrel{\chi_{1,n-1}}{\to} G^U_{0,n}$$

is a sequence of bijective maps. In short, this means that there is an bijective real correspondence (given explicitly via compositions of maps like (2.1)) between Riemannian metrics of arbitrary signature. Therefore, starting from the class of Euclidean metrics on $U \subseteq M$, we can construct all other kinds of Riemannian metrics on U by means of the maps $\chi_{p,q}$, $p + q = \dim M$. Note, in the 'smooth' case the last statement may not hold globally on M but it is always valid locally.

5. Conclusion

The main results of the previous considerations are expressed by Propositions 2.1, 4.1 and 4.2, and Corollary 2.2. Their consequence (see Corollary 4.2) is the existence of bijective mapping between metrics of signatures (p, q) and $(q + 1, p - 1)$, in particular between Euclidean and Lorentzian metrics. Another corollary of these propositions is that on a manifold exist metrics of signature $(q + 1, p - 1)$ if it admits a metric g of signature (p, q) and a vector field t with $g(t, t) > 1$. When applied to Lorentzian and Euclidean metrics, the last assertion reproduces a known result [2, Section 2.6]. Vector fields t with $g(t, t) > 1$ exist on M iff it admits a non-vanishing vector field over the whole manifold M. If we do not impose additional conditions on the last field, it always exists. But if we require it to be of class C^m with $m \geq 0$, its existence is connected with the topological properties of M

and one should explore the situation in any particular case. Generally non-vanishing C^m vector fields exist locally, but globally this may not be the case.

By Corollary 4.3 there is bijective correspondence between metrics of signature (p, q) and (q, p), etc. It is important to be noted that the case of a vector field t with $g(t, t) < 1$ significantly differs from the one of t with $g(t, t) \geq 1$ when some smoothness conditions are imposed: C^m, $m \geq 0$ vector field t with $g(t, t) < 1$ exists over any subset $U \subseteq M$, in particular over the whole manifold M. In fact, a trivial example of this kind is the zero vector field over $U \subseteq M$.

For the metrics $g^{\pm}$ and $\tilde{g}^{-}$ (see (2.4) and (2.5)) results analogous to those for $\tilde{g}^{+} := h$ in Sections 2 and 4 can be proved. Since this is an almost evident technical task, we do not present them here. In connection with this, we will note only that the equalities $(\widetilde{\tilde{g}^{\pm}})^{\pm} = g$ and $(g^{\pm})^{\mp} = g$ are valid iff $\pm g(t, t) = 0, +2$ and $\pm g(t, t) = 0, -2$, respectively, while the equations $(\widetilde{\tilde{g}^{\pm}})^{\mp} = g$ and $(g^{\pm})^{\pm} = g$ cannot be fulfilled for (real) Riemannian metrics as they are equivalent to $\pm g(t, t) = 1 - \mathrm{i}, 1 + \mathrm{i}$ and $\pm g(t, t) = -1 - \mathrm{i}, -1 + \mathrm{i}$, respectively, $\mathrm{i} := +\sqrt{-1}$.

Metrics like $g^{\pm}$, defined by (2.5), find applications in exploring modifications of general relativity. For instance, up to a positive real constant, defined in [7, Section IV, Eq. (41)] metric g^{Einst} is of the type g^{ε} with $\varepsilon = \mathrm{sign}(-\lambda)$ and $t_i = \sqrt{|2\lambda|}\eta_i$ with $\lambda := (\alpha + \beta)/(\alpha + 2\beta)$, where the real parameters α and β and the covector η_i are described in [7, Section II].

A corollary of Proposition 2.3 is the assertion of ([2], Section 2.6; [3], p. 219; [4], p. 149, Lemma 36) that if g is a Euclidean metric and X is a non-zero vector field, then $h = g - 2g(\cdot, X) \otimes g(\cdot, X)/g(X, X)$ is a Lorentzian metric. In fact, putting $t = \sqrt{2}X/\sqrt{g(X, X)}$ $(= \sqrt{2}U$ in the notation of [4]), we get $h = g - g(\cdot, t) \otimes g(\cdot, t)$ and $g(t, t) = -2 < -1$. Therefore, h has signature $(n - 1, 1)$ as that of g is $(n, 0)$, i.e., it is a Lorentzian metric according to the definition accepted in [2–4].

Since (2.1) is insensitive to the change $t \mapsto -t$, we are practically dealing with the field $(t, -t)$ of linear elements, i.e. [2, Section 2.6] a field of pairs of vector fields with opposite directions, not with the vector field t itself. If $(X, -X)$ is a field of linear elements on M, then for any $\lambda \in \mathbb{R}$, $\lambda > 1$ the vector fields $t_{\pm} := \pm\sqrt{\lambda/e(X, X)}X$ have Euclidean norm $e(t_{\pm}, t_{\pm}) = \lambda > 1$. Conversely, if t is a vector field with $e(t, t) > 1$, then $(t, -t)$ is a field of linear elements on M. Combining the just-obtained results, we infer that on M exist Lorentzian metrics iff on it exists a field of linear elements. This is a known result that can be found, e.g., in [2, Section 2.6].

Let e and h be, respectively, Euclidean and Lorentzian metrics connected by (2.1) for some t with $e(t, t) > 1$. Now we shall prove that for a suitable choice of t the set V of vector fields on M can be split into a direct sum $V = V^{+} \oplus V^{-}$ in which V^{+} is orthogonal to V^{-} with respect to both e and h, and $h|_{V^{\pm}} = \pm e|_{V^{\pm}}$. In fact, defining $V^{+} := \{t^{+} : t^{+} = \lambda t,\ \lambda \in \mathbb{R}\backslash\{0\}\}$ and $V^{-} := \{t^{-} : e(t^{-}, t) = 0\}$, we see that for $s^{\pm}, t^{\pm} \in V^{\pm}$ is fulfilled $e(t^{-}, t^{+}) = h(t^{-}, t^{+}) \equiv 0$, $h(s^{-}, t^{-}) \equiv -e(s^{-}, t^{-})$ and $h(s^{+}, t^{+}) = (e(t, t) - 1)e(s^{+}, t^{+})$. The choice of t with $e(t, t) = 2$ completes the proof. In this way we have obtained an evident special case, concerning Lorentzian metrics, of [16, p. 434, Proposition VII]. As a consequence of the last proof, as well as of (2.1), we see that any set of vector fields in V^{-} which are mutually orthogonal (or orthonormal) with respect to e is such also with respect to h for any

t with $e(t,t) > 1$ (a good choice is $e(t,t) = 2$ — see (4.1)). Sets of this kind are often used in physics [2]. Evidently, if we add to such a set the vector field t, the mutual orthogonality of the vector fields of the new set will be preserved.

In some sense, the deviation of a Lorentzian metric g from a Euclidean e can be described by an appropriate choice of certain vector field t, all connected by (2.3) under the condition $e(t,t) > 1$. In [1], this vector field is interpreted as a field of the time, the so-called temporal field. In [1], a normalization condition $e(t,t) = 1$ is imposed on t (see [1, Eq. (3)]), which, as we proved in this paper, contradicts the Riemannian character of the metrics considered. Consequently, this condition has to be dropped and replaced with $e(t,t) > 1$. The physical interpretation of t as a temporal field will be studied elsewhere.

We also have to note that the statement in [1, p. 13] that the determinants of Euclidean and Lorentzian metrics, corresponding via (2.3), differ only by sign is generally wrong. In fact, in a special basis $\{E_i\}$ in which $e_{ij} = \delta_{ij}$ and $t^i = \delta^{i1}$ hold,[16] we have $\det[g_{ij}] = (-1)^{n+1}(e(t,t) - 1)$ which in an arbitrary basis reads $\det[g_{ij}] = (-1)^{n+1}(e(t,t) - 1)\det[e_{ij}]$. Therefore, $\det[g_{ij}] + \det[e_{ij}] = 0$ is true only in two special cases, viz. if $n = 2k$ and $e(t,t) = 2$ or if $n = 2k+1$ and $e(t,t) = 0$, $k = 0, 1, \ldots$. Moreover, by Corollary 2.1, the second case cannot be realized if e is Euclidean and g Lorentzian. Thus, the mentioned statement is valid only on even-dimensional manifolds and vector fields t with norm 2.

There is a simple, but useful result for physics. Given metrics g, $g^{\pm}$, and $\tilde{g}^{\pm}$ and a vector field t non-isotropic with respect to g (i.e., $g(t,t) \neq 0$), all connected via (2.1), (2.4) and (2.5). Then there exist (local) fields of bases orthogonal with respect to all these metrics. To prove this, we notice that if $\{E_i\}$ is a field of bases with $E_n = \lambda t$, $\lambda \neq 0, \infty$ and $g(E_i, E_j) = \alpha_i \delta_{ij}$, where $\alpha_i : M \to \mathbb{R}\backslash\{0\}$ and δ_{ij} are the Kroneker δ-symbols, then $g^{\pm}(E_i, E_j) = \beta_i^{\pm}\delta_{ij}$, where $\beta_i^{\pm} = \alpha_i$ for $1 \leq i < n$ and $\beta_n^{\pm} = \alpha_n \pm \alpha_n^2/\lambda^2$, and $\tilde{g}^{\pm}(E_i, E_j) = \tilde{\beta}_i^{\pm}\delta_{ij}$ with $\tilde{\beta}_i^{\pm} = -\alpha_i$ for $1 \leq i < n$ and $\tilde{\beta}_n^{\pm} = -\alpha_n \pm \alpha_n^2/\lambda^2$.

We end with the remark that the results of this paper may find possible applications in the physical theories based on space–time models with changing signature (topology) (see, e.g. [21,22]).

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